Echelon Forms and Row Reduction

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Row-Reduced Matrix

Row-Reduced Matrix

Definition

- A $m \times n$ matrix R is called row-reduced if:
  1. Leading entries=1: The first non-zero entry in each non-zero row of R is equal to 1.
  2. All other entries in the columns corresponding to leading entries must be zero: Any column of R that contains a leading entry (the first non-zero entry) of a row must have all its other entries equal to zero.

Row-Reduced Matrix

Example

- Are following matrices Row-Reduced Matrix?

a. $n \times n$ identity matrix

b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

a. $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix}$

Example Solution!

Matrix A: Identity Matrix I_n

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ✅ Leading entries are 1
- ✅ Each leading 1 is the only nonzero entry in its column

✔ A is a row-reduced matrix.

Example Solution!

Matrix B

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

- ✅ The first leading entry (row 1) is **1** (✔).
- ❌ The second leading entry is **2 instead of 1** (**X**).
- ✅ The zero row is at the bottom (✔).

✖ **B is NOT row-reduced** because the second pivot must be **1**.

✔ **Correction:** Divide row 2 by 2 to make the leading entry **1**.

Example Solution!

Matrix C

$$C = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

✅ The leading entries are **1** (✔).

❌ The first row has **-3** in the third column, but the third column has a leading **1 in row 3** (**✗**).

✖ **C is NOT row-reduced** because the third column (with a leading 1 in row 3) has another nonzero entry **(-3 in row 1)**.

✔ **Correction:** Perform row operations to make the third column have only one nonzero entry.

Example Solution!

Matrix D

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

✅ The first leading entry is **1** in row 1 (✔).

❌ The third row has a leading **-6** instead of 1 (**✗**).

✖ **D is NOT row-reduced** because the third leading entry must be **1**.

✔ **Correction:** Divide row 3 by **-6** to make the leading entry **1**.

Row-Reduced matrix for Every Matrix

Theorem (1)

Every $m \times n$ matrix is row-equivalent to a row-reduced matrix.

Theorem 4. *Every* m $\times$ n *matrix over the field* F *is row-equivalent to a row-reduced matrix.*

Proof. Let A be an $m \times n$ matrix over F. If every entry in the first row of A is 0, then condition (a) is satisfied in so far as row 1 is concerned. If row 1 has a non-zero entry, let k be the smallest positive integer j for which $A_{1j} \neq 0$. Multiply row 1 by A_{1k}^{-1}, and then condition (a) is satisfied with regard to row 1. Now for each $i \geq 2$, add $(-A_{ik})$ times row 1 to row i. Now the leading non-zero entry of row 1 occurs in column k, that entry is 1, and every other entry in column k is 0.

Now consider the matrix which has resulted from above. If every entry in row 2 is 0, we do nothing to row 2. If some entry in row 2 is different from 0, we multiply row 2 by a scalar so that the leading non-zero entry is 1. In the event that row 1 had a leading non-zero entry in column k, this leading non-zero entry of row 2 cannot occur in column k; say it occurs in column $k_r \neq k$. By adding suitable multiples of row 2 to the various rows, we can arrange that all entries in column k' are 0, except the 1 in row 2. The important thing to notice is this: In carrying out these last operations, we will not change the entries of row 1 in columns $1, \ldots, k$; nor will we change any entry of column k. Of course, if row 1 was identically 0, the operations with row 2 will not affect row 1.

Working with one row at a time in the above manner, it is clear that in a finite number of steps we will arrive at a row-reduced matrix. ∎

02

Echelon Form

Echelon form

Definition

- A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:
  1. All nonzero rows are above any rows of all zeros.
  2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  3. All entries in a column below a leading entry are zeros.

$$\underbrace{\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix}}_{\text{Echelon form}}$$

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Row-Reduced Echelon Form

Row-Reduced Echelon form

Definition

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

1. The leading entry in each non-zero row is 1.
2. Each leading 1 is the only non-zero entry in its columns.
3. The leading 1 in the second row or beyond is to the right of the leading 1 in the row just above.
4. Any row containing only 0's is at the bottom.

$$\begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & \end{array} \\ \underbrace{\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}}_{\text{Reduced Echelon form}} \end{array}$$

Row-Reduced echelon matrix for Every Matrix

Theorem (2)

Every $m \times n$ matrix is row-equivalent to a row-reduced echelon matrix.

Proof is same as the Theorem (1).

Reduced Echelon Form (RREF)

Example

Are following matrices RREF?

a. $0_{m\times n}$

b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 1 & -3 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Example Solution!

Matrix A: Zero Matrix $0_{m\times n}$

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

✅ All entries are zero, so **it trivially satisfies RREF conditions.**

✔ A is in RREF.

Example Solution!

Matrix B

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

✅ The leading 1 in the first row is correct.

❌ The second row has a leading **2 instead of 1**.

✖ **B is NOT in RREF** because the second pivot must be **1**.

✔ **Correction:** Divide row 2 by **2** to make the pivot **1**.

Example Solution!

Matrix C

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

❌ The first nonzero entry in the first row is **not in the first column.**

❌ The first nonzero entry in the second row is **before the one in the first row**, which **violates echelon** form.

✖ C is NOT in RREF.

✔ **Correction:** Swap rows to make leading 1s in proper positions.

Example Solution!

Matrix D

$$D = \begin{bmatrix} 0 & 1 & -3 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- ✅ Leading 1 in row 1 is **in the correct position.**
- ✅ Leading 1 in row 2 is **to the right of the leading 1 in row 1.**
- ✅ The zero row is at the bottom.
- ✅ No other nonzero entries in pivot columns.

✔ **D is in RREF!**

Existence and Uniqueness Questions

Two fundamental questions about a linear system:

1. Is the system consistent? That is, does at least one solution exist?
2. If a solution exists, is it the only one? That is, is the solution unique?

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Solutions of a Linear System

Elementary Row Operations

Example

- Augmented matrix for a linear system:

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x - 5z &= 1 \\ y + z &= 4 \\ 0 &= 0 \end{aligned} \qquad \begin{cases} x = 1 + 5z \\ y = 4 - z \\ z \text{ is free variable} \end{cases}$$

- x, y: basic variable $\qquad$ z: free variable
- This system is consistent, because the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables.

Existence and Uniqueness Questions

Theorem (3)

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the augmented matrix has no row of the form $[0 \quad \cdots \quad 0 \quad b]$ with nonzero b.

- b is any nonzero value such as $[0 \quad \cdots \quad 0 \quad -2], [0 \quad \cdots \quad 0 \quad 20], \ldots.$
- Let's think about this proof in next slide!

Existence and Uniqueness Questions

Forward Direction: If the system is consistent, the rightmost column is not a pivot column.

- A system is consistent if at least one solution exists.
- To solve the system, we reduce the augmented matrix to row echelon form.
- If a row of the form $\begin{bmatrix} 0 & 0 & \dots & 0 & b \end{bmatrix}$ (with $b \neq 0$) appears, it represents the equation:

$$0 = b, \quad \text{which is a contradiction!}$$

- This means the system is **inconsistent**.
- Therefore, if the system is **consistent**, such a row **must not exist**.
- This implies that the **rightmost column** cannot be a **pivot column**, since having a pivot in the last column would force the existence of such a contradictory row.

Existence and Uniqueness Questions

Reverse Direction: If the rightmost column is not a pivot column, the system is consistent.

If the rightmost column **is not a pivot column**, then **no row** of the form $[0 \quad 0 \quad \dots \quad 0 \quad b]$ (where $b \neq 0$) appears in echelon form.

This means that no contradiction arises when solving the system.

Therefore, at least one solution exists, and the system is **consistent**.

A linear system is consistent $\Leftrightarrow$ The rightmost column of the augmented matrix is not a pivot column.

- If a linear system is consistent, then the solution set contains either:
  - A unique solution, when there are no free variables
  - Infinitely many solutions, when there is at least one free variable

Find all solutions of a linear system

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Existence of Solutions

Example

Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $Ax = b$ consistent for all possible b_1, b_2, b_3?

Solution

Change the augmented matrix for $Ax = b$ to echelon form:

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

The third entry in column 4 equals $b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1)$. The equation $Ax = b$ is not consistent for every b because some choices of b can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero.

Existence of solutions

Example

True or False?

Equation $Ax = b$ is consistent, if its augmented matrix $[A\ b]$ in RREF has one pivot column in each rows? (Having one leading entry in each rows)

🚫 **False!**

If the augmented matrix has **one pivot in each row**, it does **not guarantee** consistency. Consider:

$$\begin{bmatrix} 1 & 2 & |3 \\ 0 & 1 & |5 \end{bmatrix}$$

✅ **Consistent (No pivot in last column)**

But in this case:

$$\begin{bmatrix} 1 & 2 & |3 \\ 0 & 0 & |5 \end{bmatrix}$$

❌ **Inconsistent!** (Pivot in last column → $0 = 5$ contradiction)

Homogeneous Linear Systems

Definition

A system of linear equations is said to be homogeneous if it can be written in the form $Ax = 0$, where A is a matrix and 0 is the zero vector.

- Trivial solution: $Ax = 0$ always has at least one solution, namely, $x = 0$ (the zero vector)
- Nontrivial solution: The non-zero solution for $Ax = 0$.

Fact

The homogenous equation $Ax = 0$ has a nontrivial solution if and only if the equation has at least one free variable.

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Proof

Suppose A is an $m×n$ matrix. After reducing A to RREF, there are two possibilities:

1. All variables are pivot variables: Each variable is leading in a row → only solution is x=0 (trivial).
2. At least one free variable: Let xj be free. Set xj=t≠0, solve for the pivot variables → nonzero solution exists.

Existence Of Solutions

Fact

The equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A.

Part 1 – "If" Direction

Let $A \in R_{m \times n}$ with columns $a_1, \dots, a_n$.
Write Ax=b:

$$Ax = a_1 x_1 + \cdots + a_n x_n$$

If Ax=b, then:

$$b = a_1 x_1 + \cdots + a_n x_n$$

☑ b is a linear combination of columns of A.

Part 2 – "Only If" Direction

Suppose b is a linear combination of columns of A:

$$b = a_1 x_1 + \cdots + a_n x_n$$

Matrix form:
$b = Ax$, with $x = [x_1, \dots, x_n]^T$
☑ Ax=b has a solution.

05

Geometric Interpretation

Line (R^2)

The line ℓ with equation $2x + y = 0$

$\mathbf{n} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, then the equation becomes $\mathbf{n} \cdot \mathbf{x} = 0$.

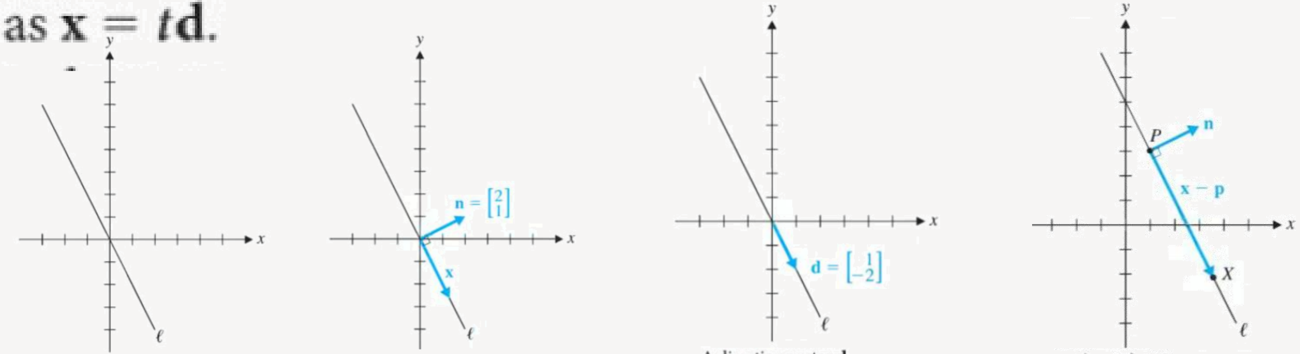
ℓ as $\mathbf{x} = t\mathbf{d}$.

Definition The *normal form of the equation of a line* ℓ in $\mathbb{R}^2$ is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0 \quad \text{or} \quad \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$$

where $\mathbf{p}$ is a specific point on ℓ and $\mathbf{n} \neq \mathbf{0}$ is a normal vector for ℓ.

The *general form of the equation of* ℓ is $ax + by = c$, where $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a normal vector for ℓ.



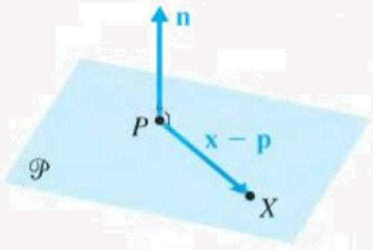
The line $2x + y = 0$ | A normal vector $\mathbf{n}$ | A direction vector $\mathbf{d}$ | $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$

Lines in $\mathbb{R}^2$			
Normal Form	**General Form**	**Vector Form**	**Parametric Form**
$\boldsymbol{n} \cdot \boldsymbol{x} = \boldsymbol{n} \cdot \boldsymbol{p}$	$ax + by = c$	$\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{d}$	$\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \end{cases}$

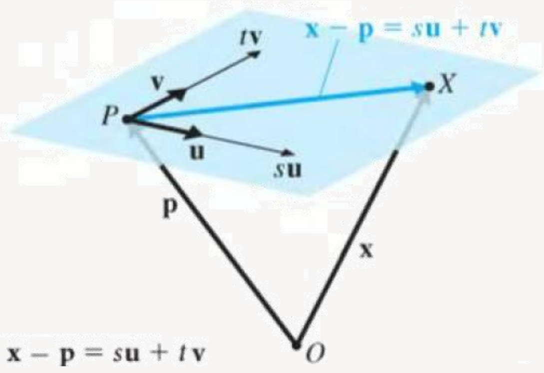
Plane (R^3)

$$\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$ax + by + cz = d$ (where $d = \mathbf{n} \cdot \mathbf{p}$)



$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$



$\mathbf{x} - \mathbf{p} = s\mathbf{u} + t\mathbf{v}$

Lines and Planes in $\mathbb{R}^3$

	Normal Form	General Form	Vector Form	Parametric Form
Lines	$\begin{cases} \boldsymbol{n_1} \cdot \boldsymbol{x} = \boldsymbol{n_1} \cdot \boldsymbol{p_1} \\ \boldsymbol{n_2} \cdot \boldsymbol{x} = \boldsymbol{n_2} \cdot \boldsymbol{p_2} \end{cases}$	$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$	$\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{d}$	$\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \\ z = p_3 + td_3 \end{cases}$
Planes	$\boldsymbol{n} \cdot \boldsymbol{x} = \boldsymbol{n} \cdot \boldsymbol{p}$	$ax + by + cz = d$	$\boldsymbol{x} = \boldsymbol{p} + s\boldsymbol{u} + t\boldsymbol{v}$	$\begin{cases} x = p_1 + su_1 + tv_1 \\ y = p_2 + su_2 + tv_2 \\ z = p_3 + su_3 + tv_3 \end{cases}$

Nonhomogeneous Systems & General Solution

Question

Can we change the order of columns in an augmented matrix???

$$\begin{cases} ax & + & by & + & cz & = & d \\ a'x & + & b'y & + & c'z & = & d' \\ a''x & + & b''y & + & c''z & = & d'' \end{cases}$$

Is equivalent to

$$\begin{cases} ax & + & cz & + & by & = & d \\ a'x & + & c'z & + & b'y & = & d' \\ a''x & + & c''z & + & b''y & = & d'' \end{cases}$$

STUDENTIME

Time to Practice on Your Own! 🚀

Nonhomogeneous Systems & General Solution

Example

Describe all solutions of $A\boldsymbol{x} = \boldsymbol{b}$, where: $A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$

$$x = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example

Describe all solutions of $A\boldsymbol{x} = \boldsymbol{0}$, where: $A = \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

$$x = s \begin{bmatrix} 8 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Example

Describe all solutions of $A\boldsymbol{x} = \boldsymbol{b}$, where: $[A\ b] = \begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 & 5 \end{bmatrix}$.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Resources

- Chapter 1: Kenneth Hoffman and Ray A. Kunze. Linear Algebra. PHI Learning, 2004.
- Chapter 1: David C. Lay, Steven R. Lay, and Judi J. McDonald. Linear Algebra and Its Applications. Pearson, 2016.
- Chapter 2: David Poole, Linear Algebra: A Modern Introduction. Cengage Learning, 2014.
- Chaper1: Gilbert Strang. Introduction to Linear Algebra. Wellesley-Cambridge Press, 2016.